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14. ABSTRACT

In section 1, we describe our results on optimal control of a long term average cost rate problem. A new method, "an Abelian limit approach" is developed to solve this singular control problem. Abelian limit relations between three types of value functions are also developed. Results on two singular control problems arising from queueing networks in heavy traffic are described in section 2. There we derive an "optimal buffer size" for the singular control problem and it will yield asymptotically optimal buffer lengths for the corresponding queueing networks in heavy traffic.

In section 3, the results on a combined control and stopping problem is described. Novelty feature here is that the diffusion coefficient is controlled and it is allowed to take the value zero. A martingale characterization is derived. Also, an explicit optimal strategy is obtained when the drift coefficient is linear. A stochastic differential game with degenerate variance control is described

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Final Progress Report

Contents

Controlled stochastic dynamical systems

1 Singular Control

1.1 Problem formulation

The work described in this section is related to section 3 of the original proposal [21]. Consider a weak solution to the one-dimensional stochastic differential equation

$$X_x(t) = x + \int_0^t \mu(X_x(s-))ds + \int_0^t \sigma(X_x(s-))dW(s) + A(t) \tag{1.1}$$

where x is a real number, $\{W(t) : t \geq 0\}$ is a standard Brownian motion adapted to a right continuous filtration $\{\mathfrak{F}_t : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. The functions μ and σ are differentiable and satisfy the assumptions listed below. The control process A(.) is $\{\mathfrak{F}_t\}$-adapted, right continuous with left limits, and of bounded variation on finite time intervals. Also, A(0)=0.

The main assumption here is that the drift and diffusion coefficients satisfy the following two conditions:

The functions μ and σ are continuously differentiable on $\mathbf{R}$, $\mu'(y) \leq 0$ for all y, $\inf_{\mathbf{R}} \sigma(y) > 0$ and $x\mu(x) < 0$ for all $x \neq 0$. (1.2)

The diffusion coefficient σ is allowed to be unbounded subjected to growth conditions

$$\int_{-\infty}^0 \frac{\mu(x) - x}{\sigma^2(x)} dx = \int_0^\infty \frac{x - \mu(x)}{\sigma^2(x)} dx = \infty. \tag{1.3}$$

In the absence of the control process, (1.1) can be considered as a random perturbation of the stable dynamical system $\dot{x} = \mu(x)$ which has a unique global, asymptotically stable equilibrium point at the origin. The cost function $C(\cdot)$ also has its unique minimum at the origin and it increases as x moves away from the origin.Therefore, our study concerns the long term stability of a randomly perturbed stable dynamical system with a minimal control effort. Such situations are abundant in applications, such as Capacity Expansion Theory(see example 2, p.393 in [23], [19]).

Let $\mathcal{U}$ be the collection of such processes (X_x, A) which satisfy (1.1) together with above assumptions.We shall study the ergodic stochastic control

problem with optimal value λ_0 defined by

$$\lambda_0 \triangleq \inf_{\mathcal{U}} \limsup_{T\to\infty} \frac{1}{T} E_x \int_0^T [C(X_x(s))ds + d|A|(s)]. \tag{1.4}$$

Notice that λ_0 is a constant which is independent of the initial value x. In [25], we were able to characterize an optimal control with a Markovian state process X that achieves the value λ_0 and to relate it to the value functions of the family of discounted control problems defined by

$$V_\alpha(x) \equiv \inf_{\mathcal{U}} E_x \int_0^\infty e^{-\alpha s}[C(X_x(s))ds + d|A|(s)] \tag{1.5}$$

as well as to the value functions of the family of finite horizon control problems

$$V_0(x,T) \equiv \inf_{\mathcal{U}} E_x \int_0^T [C(X_x(s))ds + d|A|(s)]. \tag{1.6}$$

The qualitative nature of the optimal policies depends on the growth rates of $|\mu'(x)|$ and $|C'(x)|$. Therefore, we introduce a function H defined by

$$H(x) = \mu'(x) + |C'(x)| \qquad \text{for all } x \text{ in } \mathbf{R}. \tag{1.7}$$

We were also able to apply our results of ergodic control problem to derive a complete solution for the following minimization problem with a constraint:
Let $m > 0$ be a constant and let $\mathcal{U}_m$ be the collection of processes (X_x, A) in $\mathcal{U}$ which satisfy the constraint

$$\limsup_{T\to\infty} \frac{E|A|(T)}{T} \le m. \tag{1.8}$$

The constrained problem is the following:

$$\text{Minimize} \qquad \limsup_{T\to\infty} \frac{1}{T} E \int_0^T C(X_x(s))ds \tag{1.9}$$

over all the admissible processes (X_x, A) in $\mathcal{U}_m$.

1.2 Main results and their importance

In [25], we were able to obtain the following results.

1. If $H(x) \leq 0$ for all x, then zero control policy is optimal.

2. If there exist constants $\epsilon > 0$ and $\delta > 0$ such that $H(x)+\epsilon\mu'(x) > \delta > 0$ for large $|x|$, then there is a finite interval $[a^*, b^*]$ such that the optimal state process X_x^* is a reflecting diffusion on $[a^*, b^*]$. In this case, optimal control process A^* is the difference of local-time processes of X_x^* at the points a^* and b^*.

3. In the both cases above, the following Abelian limits hold:
$\lim_{\alpha \to 0} \sup_{|x| \leq K} |\alpha V_\alpha(x) - \lambda_0| = 0 \quad \text{and} \quad \lim_{T \to \infty} \sup_{|x| \leq K} \left| \frac{V_0(x,T)}{T} - \lambda_0 \right| = 0$
for each $K > 0$.

4. There is a constant $m_0 > 0$ such that for all $m > m_0$, the constrained optimization problem described in (1.8) and (1.9) has an optimal strategy of the sort described in item 2 above.

Our results in items 1 and 2 greatly generalizes the works of [9] and [14]. Items 1 and 2 provides a near dichotomy of very different types of optimal strategies on how to control a random perturbation of a stable dynamical system. Unlike in many other articles in the literature, we make no assumptions on the convexity of the running cost function. Our approach to ergodic control problem is completely new and we use Abelian limits and this method can be applied in other situations. In an interesting article [1], Arisawa and Lions obtained the uniform convergence of the Abelian limits in item 3 (uniformly on the state space), when the state process is in a compact space. Our results generalize their work to processes on whole real line and our work shows that uniform convergence on compact sets is the best possible when the state space is non-compact.

In a recent article, [2], Ata et al. considered a constrained minimization problem similar to (1.8) and (1.9) when the state space is a fixed interval $[0, b]$. Their work arises from an application in Wireless network communication theory. Our work in section 6 of [25] generalizes those results to a non compact state space. For constrained optimization problems with dynamic constraints, new methods were developed in [3].

2 Singular Control and Queues in Heavy Traffic

The work described in this subsection is the result of a joint project with Arka P. Ghosh in the Statistics Dept., Iowa State University.

Optimal buffer size for queueing networks is a very important design consideration in engineering applications and there are very few technical results available in the literature(see [2]). The following two applications of singular control theory address this issue when the queueing networks are in heavy traffic.

2.1 Optimal Buffer Length Problem

We consider a one dimensional singular control problem which can be obtained as a formal limit of a diffusion scaled queueing network models. The non-negative state process X_x which represents the queue length process is given by

$$X_x(t) = x - \int_0^t u(s)ds + W(t) + L(t) - U(t) \tag{2.1}$$

where $W(\cdot)$ is a one dimensional Brownian motion adapted to a right continuous filtration $\{\mathfrak{F}_t : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. The process $L(\cdot)$ is the local-time process of the state process X_x at the origin.
The processes $u(\cdot)$ and $U(\cdot)$ are considered control processes and are adapted to the filtration $\{\mathfrak{F}_t : t \geq 0\}$. The drift control process $u(\cdot)$ takes values in an a priori known control set $\mathcal{A}$ and this process is analogous to the difference between the service rate and the arrival rate of the queueing model.It is also assumed that $E\int_0^t u(s)ds < \infty$ for each $t > 0$.
The other control process $U(\cdot)$ is a non-decreasing RCLL process and $U(0) = 0$. In the queueing model, controller is allowed to reject the customers who try to enter the queue to keep the congestion cost low. In (2.1), $U(t)$ represents the cumulative number of "rejected" customers during $[0, t]$. In general, there are no constraints on the queue length, but the controller may choose a threshold level $b > 0$ for the queue length to control the costs.
We assume that the cost functions h and C are non-negative, convex C^2 functions and they represent the congestion cost(or holding cost) and the control cost as described below. For a given admissible control system (X_x, u, U), we introduce the cost functional $\zeta(T)$ by

$$\zeta(T) = \int_0^T [h(X_x(s)) + C(u(s))]ds + p \cdot U(T) \tag{2.2}$$

Here, p is a positive constant which represents the penalty per rejection. The long term average cost per unit time is represented by the cost functional

$$J(X_x, u, U) = \limsup_{T\to\infty} \frac{1}{T} E[\zeta(T)]. \tag{2.3}$$

The value function of this ergodic problem is given by

$$v_0 = \inf_{(X_x,u,U)} J(X_x, u, U) \tag{2.4}$$

The problem addressed in [7] is to find an optimal control policy (X_x^*, u^*, U^*), for the control problem (2.4) under reasonable assumptions and to characterize the value v_0 in (2.4).
In addition to the previous assumptions, we also assume that the control set $\mathcal{A}$ is a subset of $[0, \infty)$ and it contains the interval $[0, \theta_0]$ where θ_0 satisfies $C'(\theta_0) = 1$.

2.2 Main results on optimal buffer size and their importance

1. We were able to show that there is an optimal buffer length b^* related to the optimal policy. The controller should not let the queue-length to exceed this threshold point b^*. More precisely, we obtained a Markovian optimal policy described by

$$X_x^*(t) = x - \int_0^t u^*(X_x^*(s))ds + W(t) + L^*(t) - U^*(t) \tag{2.5}$$

   Here, the optimal drift control $u^*(X_x^*(t))$ is feed-back type and the optimal rejection process $U^*(t)$ is the local-time process of X_x^* at the threshold b^*. L^* is the local-time process of X_x^* at the origin.

2. The value v_0 is characterized by an equation which relates v_0 with the threshold b^*.The existence of a unique C^2 solution to the corresponding (HJB) equation is also established in this work.

3. An algorithm is developed to compute the threshold b^*.

   When the buffer size is fixed, and there is no congestion cost(i.e. $h(\cdot)$ is identically zero), this problem reduces to the work of [2]. Also,the article [6], address a discrete-time control problem for a queueing system in equilibrium and their cost structure contains a control cost and

a congestion cost, but there is no rejection cost. Our work is motivated by these two articles and our cost structure incorporates all three types of costs. The optimal policy derived here can be used to obtain asymptotically optimal policies for large queueing networks in heavy traffic. For a discussion on the use of optimal singular control to find asymptotically optimal control policies for queueing networks in heavy traffic, we refer to [20].

2.3 Dynamic control of a queueing network with reneging

This is a project [8] which is just completed and currently we are in the process of preparing a first draft. This work is motivated by our previous work in [7] and also by the recent articles of [18] and [20]. The model here is similar to that of (2.1) but in addition to that impatient customers are allowed to leave the queue at a rate of $\gamma > 0$. Therefore with the same notation as in (2.1), the state process X_x satisfies the equation

$$X_x(t) = x - \int_0^t u(s)ds - \gamma \int_0^t X_x(s)ds + W(t) + L(t) - U(t) \tag{2.6}$$

for all $t > 0$.

The control processes u and U satisfies the same assumptions as in (2.1)and the drift control process $u(\cdot)$ takes values in an a priori known control set $\mathcal{A}$. But in this case, we address the infinite horizon discounted control problem. To describe it, given an admissible control policy (X_x, u, U), the discounted cost $J_p(X_x, u, U)$ is given by

$$J_p(X_x, u, U) = \int_0^\infty e^{-\delta s}[(\beta\gamma X_x(s) + C(u(s)))ds + p \cdot dU(s)] \tag{2.7}$$

where β and δ are positive constants. Instead of the congestion cost $h(\cdot)$ term in (2.2), here there is a reneging cost due to the departure of the impatient customers. As in (2.2), the constant p represents the penalty rate per rejection. The value function is defined by

$$V_p(x) = \inf_{(X_x,u,U)} J_p(X_x, u, U) \tag{2.8}$$

for each $x \geq 0$.

Next we introduce the constant $p_0 > 0$, which is crucial in the determination of our optimal control policy. Let

$$p_0 = \frac{\beta\gamma}{\delta + \gamma} \tag{2.9}$$

where the constants β, γ and δ are as above. Our main assumption here is that the control set $\mathcal{A}$ is a subset of $[0, \infty)$ and it contains the interval $[0, \theta_0]$ where θ_0 is the unique positive real number which satisfies

$$C'(\theta_0) = p_0. \tag{2.10}$$

2.4 Main results and their importance

1. When $0 < p < p_0$, we show that there is a Markovian optimal policy and a corresponding threshold $b^* > 0$ described by

$$X_x^*(t) = x - \int_0^t u^*(X_x^*(s))ds - \gamma \int_0^t X_x^*(s)ds + W(t) + L^*(t) - U^*(t) \tag{2.11}$$

   Here, the optimal drift control $u^*(X_x^*(t))$ is feed-back type and the optimal rejection process $U^*(t)$ is the local-time process of X_x^* at the threshold b^*. L^* is the local-time process of X_x^* at the origin.
   Therefore, the threshold b^* can be considered as the optimal buffer length associated with (2.8).

2. When $p \geq p_0$, we show that there is an optimal Markovian policy and the state process X_x^* can be described by

$$X_x^*(t) = x - \int_0^t u^*(X_x^*(s))ds - \gamma \int_0^t X_x^*(s)ds + W(t) + L^*(t) \tag{2.12}$$

   Here L^* is the local-time process of X_x^* at the origin and there is no rejection at all and thus the optimal rejection process U^* is identically zero. Therefore, the optimal state process X_x^* takes values in $[0, \infty)$ and thus there is no optimal finite buffer size.

3. The state process of the singular control problem is a diffusion limit of a sequence of state processes of queueing networks in heavy traffic. Therefore, as similar to [20], we use our explicit optimal policies described above to obtain asymptotically optimal policies for the corresponding queueing network control problems.

3 Discretionary stopping and degenerate variance control

3.1 Problem formulation

The results in this section is joint work with Daniel Ocone, Dept. of Mathematics, Rutgers Univ., New Brunswick. This work is related to the proposed problems in section 2 of the original proposal [21].

Consider a one dimensional stochastic differential equation

$$X_x^u(t) = x + \int_0^t b(X_x^u(s))ds + \int_0^t u(s)dW(s) \tag{3.1}$$

where x is a real number, $\{W(t) : t \geq 0\}$ is a standard one-dimensional Brownian motion adapted to a filtration $\{\mathfrak{F}_t : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. The control process $u(\cdot)$ is progressively measurable with respect to the above filtration and the controller is allowed to choose it subject to the constraint

$$0 \leq u(t) \leq \sigma_0 \quad \text{for all} \quad t \geq 0. \tag{3.2}$$

Here σ_0 is a given positive constant.

We assume the following:

1. The drift coefficient $b(\cdot)$ is a C^1-function, $b(0) = 0$ and the origin is the unique asymptotically stable equilibrium point for the differential equation $\frac{dx}{dt} = b(x)$ and

2. the running cost function $C(\cdot)$ is a strictly concave C^2-function which attains its unique maximum at the origin.

Given an adapted variance control process u and a stopping time τ with respected to the filtration $\{\mathfrak{F}_t : t \geq 0\}$,we call (u, τ) an admissible control policy and let $\mathcal{U}$ be the collection of all admissible control policies. We define the reward functional

$$J(x, u, \tau) = E \int_0^\tau e^{-\alpha t} C(X_x^u(t))dt \tag{3.3}$$

where α is a positive constant. Hence, the value function is given by

$$V(x) = \sup_{\mathcal{U}} J(x, u, \tau). \tag{3.4}$$

3.2 Main results and their importance

Control problems with discretionary stopping are addressed in the recent literature([10],[12], [13]). In these articles, only the drift coefficient is allowed to control and the diffusion coefficient is strictly non-degenerate. Importance of our work is due to the following three reasons.

a) The diffusion coefficient is controlled here while all the previous articles address only the drift control problems.

b) We allow the controlled diffusion coefficient to take the value zero.

c) We provide a martingale characterization (as similar to [12]) for the value function for the degenerate variance control problem.

The state process can evolve according to a combination of a deterministic and stochastic behavior. We obtain an optimal strategy when the drift term has the linear form $b(x) = -\theta x$ where θ is a positive constant. Here are the main results:

1. Let $Q(\cdot)$ be a non-negative, bounded continuous function defined on $\mathbf{R}$ and let the initial point x be fixed.

   (i) If $Q(X_x^u(t\wedge\tau))e^{-\alpha(t\wedge\tau)}+\int_0^{t\wedge\tau} e^{-\alpha t}C(X_x^u(t))dt$ is a super-martingale for each admissible control policy (u, τ) in $\mathcal{U}$, then $Q(x) \geq V(x)$.

   (ii) If $Q(\cdot)$ satisfies the above condition (i) and if there is a state process $Z_x^{u^*}(\cdot)$ corresponding to an admissible control policy (u^*, τ^*) so that $Q(Z_x^{u^*}(t \wedge \tau^*))e^{-\alpha(t\wedge\tau^*)} + \int_0^{t\wedge\tau^*} e^{-\alpha t}C(Z_x^{u^*}(t))dt$ is a martingale and $Q(Z_x^{u^*}(\tau^*)) = 0$ on the set $[\tau^* < \infty]$, then $Q(x) = V(x)$ and $Z_x^{u^*}(\cdot)$ is an optimal state process and (u^*, τ^*) is the corresponding optimal control policy.

2. When $b(x) = -\theta x$ where θ is a positive constant, the value function $V(\cdot)$ in (3.4) is a C^1-function. Furthermore, there exist four points $c^* < \alpha^* < 0 < \beta^* < d^*$ such that the following admissible strategy (u^*, τ^*) is optimal.

   (a) If $x \leq c^*$ or $x \geq d^*$ then choose $\tau^* = 0$ and stop.

   (b) If $\alpha^* \leq x \leq \beta^*$, then choose $\tau^* = \infty$, $u^*(t) = 0$ for all t and follow the deterministic motion.

(c) If $\beta^* < x < d^*$ then choose $u^*(t) = \sigma_0$ and let $\hat{\tau}$ be the first exit time of the process $X_x^{u^*}(\cdot)$ from the interval (β^*, d^*). Thereafter, follow as in the steps a) or b) appropriately.
In this case, $\tau^* = \hat{\tau} I_{[X_x^{u^*}(\hat{\tau})=d^*]} + \infty \cdot I_{[X_x^{u^*}(\hat{\tau})=\beta^*]}$.

(d) If $c^* < x < \alpha^*$ then choose $u^*(t) = \sigma_0$ and let $\hat{\tau}$ be the first exit time of the process $X_x^{u^*}(\cdot)$ from the interval (c^*, α^*). Thereafter, follow as in the steps a) or b) appropriately.
In this case, $\tau^* = \hat{\tau} I_{[X_x^{u^*}(\hat{\tau})=c^*]} + \infty \cdot I_{[X_x^{u^*}(\hat{\tau})=\alpha^*]}$.

4 Controller-and-stopper game with degenerate variance control.

4.1 Problem formulation

The work described in this section is related to the proposed problems in section 4 of the proposal [21]. Our results are published in [24].
We consider a stochastic differential game which involves two players, the controller and the stopper. The stopper selects the stopping rule which halts the game, while the controller chooses the diffusion coefficient of the state process. Our model is described by (3.1) and the controlled diffusion coefficient satisfies (3.2). At the end of the game, the controller pays the stopper, the amount $\int_0^{\tau(X_x)} e^{-\alpha t} C(X_x(t))dt$ where α is a positive discount factor. We assume that the drift coefficient b and the reward function C satisfy the assumptions 1 and 2 in the previous section. In addition, we assume that $\alpha - b'(x) > 0$ for all x.
Let $\mathcal{S}$ be the collection of all stopping rules and $\mathcal{A}(x)$ be the collection of all controlled state processes which satisfy (3.1). We define the upper and lower value functions of this game by

$$\overline{V}(x) = \inf_{\tau \in \mathcal{S}} \sup_{X_x \in \mathcal{A}(x)} E \int_0^\tau e^{-\alpha t} C(X_x(t))dt \tag{4.1}$$

and

$$\underline{V}(x) = \sup_{X_x \in \mathcal{A}(x)} \inf_{\tau \in \mathcal{S}} E \int_0^\tau e^{-\alpha t} C(X_x(t))dt \tag{4.2}$$

respectively.

If $\overline{V}(x) = \underline{V}(x)$, then this game has a *value*, and in that case, we denote this common value function by $V(x)$. A pair (τ^*, Z^*) in $\mathcal{S} \times \mathcal{A}(x)$ is called

a *saddle point* of the game, if

$$\begin{aligned} E\int_0^{\tau} e^{-\alpha t}C(Z^*(t))dt &\le E\int_0^{\tau^*(Z^*)} e^{-\alpha t}C(Z^*(t))dt \\ &\le E\int_0^{\tau^*(X)} e^{-\alpha t}C(X(t))dt \end{aligned} \tag{4.3}$$

for every τ in $\mathcal{S}$ and every $X(\cdot)$ in $\mathcal{A}(x)$.

The existence of a saddle point clearly implies that the game has a value and in this case,

$$\overline{V}(x) = \underline{V}(x) = E\int_0^{\tau^*(Z^*)} e^{-\alpha t}C(Z^*(t))dt.$$

We intend to characterize a saddle point and to derive the explicit form of the value function for this game.

4.2 Main results and their importance

Two player, zero sum, stochastic differential games are considered in [5] using the viscosity solutions framework. They show that the existence of the value function when the processes are non degenerate and the drift and diffusion coefficients are bounded. When the diffusion coefficient is strictly positive and only when the drift is controlled, the controller-and-stopper game is analyzed in [11]. We refer to this article for the history of the discrete-time game and references. In a recent manuscript [13], a martingale formulation is developed for this game with only drift control. In [24],we treat the problem when the diffusion coefficient is controlled and allowed to degenerate. Our work use the methods we developed for the degenerate control problems in [16] and [17]. Here we derive the value function and characterize its smoothness. We describe the main results below:

1. The stochastic differential game defined in (4.1) and (4.2) has a value. The corresponding saddle point is characterized by the existence of two points $a^* < 0 < b^*$. Let the exit time of any state process X_x to be given by the stopping time $\tau^*(X_x)$ and the state process Z_x^* satisfies (3.1) with the feed-back type diffusion control $u^*(t) = \sigma_0 I_{(a^*,b^*)}(Z_x^*(t))$. Then (τ^*, u^*) forms a saddle point.

2. The value function is Lipschitz continuous but it fails to be a C^1-function.

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6 Publications supported under this grant

A. Papers in peer-reviewed journals.

1. A Controller and a Stopper Game with Variance Control for a class of Diffusion Processes, Elec. Comm. in Probab., Vol. 11,(2006), 89-99.

2. Optimal Buffer size for a Stochastic Processing Network (with Arka Ghosh) (2006) Accepted to Queueing Systems Journal.(Already appeared online in the journal's website on March 23, 2007)

3. An Abelian Limit Approach for a Singular Ergodic Control Problem of Diffusion Processes,(Single authored, 2007), Accepted to SIAM J. control Opt.(in the press. Electronic version will appear in the journal's website soon.) 40 pages

B. Manuscript submitted, but not pulished yet.

1. A Degenerate Variance Control Problem with Discretionary Stopping, (with Dan Ocone),(2006) Submitted to Thomas G. Kurtz Fetschrift. (Institute of Mathematical Statistics, peer reviewed publication).

C. Manuscript in preparation.

1. Dynamic control of queueing networks with reneging in heavy traffic, (Jointly with Arka Ghosh, Currently we are typesetting this article of 40 pages, approx.) 2007.